DNIB, 2A

Corrigé du devoir surveillé du 26.10.00

Exercice 1

On a (cf cours) $\tan u = u + \frac{u^3}{3} + \frac{2}{15}u^5 + O(u^7)$

Avec $u = x^2$ (ce qui est justifié puisque $\lim\limits_{x\to 0} u = 0$), on a

$$\tan(x^2) = x^2 + \frac{x^6}{3} + \frac{2}{15}x^{10} + O(x^{14}).$$

D'autre part, $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \frac{u^5}{5} + O(u^6)$ au vois. de 0.

On peut poser $u = \tan(x^2)$ puisque $\lim\limits_{x\to 0} u = 0$.

D'où $\ln(1-\tan x^2) = -\left(x^2 + \frac{x^6}{3} + \frac{2}{15}x^{10}\right) - \frac{1}{2}\left(x^4 + \frac{2}{3}x^8\right) - \frac{1}{3}\left(x^6 + x^{10}\right)$

$$-\frac{1}{4}x^8 - \frac{1}{5}x^{10} + O(x^{12})$$

$$\ln(1-\tan x^2) = -x^2 - \frac{1}{2}x^4 - \frac{2}{3}x^6 - \frac{7}{12}x^8 - \frac{2}{3}x^{10} + O(x^{12}).$$

Exercice 2

- On a $0 \le |f(x)| \le |x| \quad \forall x > 0$, donc $\lim\limits_{x\to 0} f(x) = 0$.
- On a $\forall x \neq 0$, $g(x) = 0$, donc $\lim\limits_{\substack{x\to 0\\(x\neq 0)}} g(x) = 0 \neq g(0)$.

Donc g n'est pas continue en 0.

- $$g\circ f(x) = \begin{cases} 0 \text{ si } f(x) \neq 0 \\ 1 \text{ si } f(x) = 0 \end{cases} = \begin{cases} 0 \text{ si } x \neq \frac{1}{n\pi} \ (n\in\mathbb{N}^*),\ \underline{x>0}. \\ 1 \text{ si } x = \frac{1}{n\pi} \quad n\in\mathbb{N}^* \end{cases}$$

On a deux suites : • $u_n = \frac{1}{n\pi}$, t.q. $g\circ f(u_n) = 1 \quad \forall n \geq 1$.

• $v_n = \frac{2}{(2n+1)\pi}$, tq $g\circ f(v_n) = \frac{(-1)^n 2}{(2n+1)\pi}$. ⟶

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = 0 \text{ , mais}$$

$$\lim_{n\to+\infty} g\circ f(u_n) = 1 \quad \text{et} \quad \lim_{n\to+\infty} g\circ f(v_n) = 0 .$$

$g\circ f$ n'admet donc pas de limite en 0.

Exercice 3.

1. Posant $u = e^x$ on obtient

$$I = \int_{\frac{-1+\sqrt{3}}{2}}^{1} \frac{du}{u\,(u^2+u+1)}$$

mais $\displaystyle \frac{1}{u(u^2+u+1)} = \frac{1}{u} - \frac{u+1}{u^2+u+1} = \frac{1}{u} - \frac{1}{2}\frac{2u+1}{u^2+u+1} - \frac{2}{3}\frac{1}{1+\left(\frac{2u+1}{\sqrt{3}}\right)^2}$

donc

$$I = \left[\ln|u| - \frac{1}{2}\ln\left(u^2+u+1\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u+1}{\sqrt{3}}\right)\right]_{\frac{-1+\sqrt{3}}{2}}^{1}$$

$$I = -\frac{1}{2}\ln 2 - \ln\left(\frac{-1+\sqrt{3}}{2}\right) - \frac{\pi\sqrt{3}}{36} = \ln\left(\frac{\sqrt{2}}{\sqrt{3}-1}\right) - \frac{\pi\sqrt{3}}{36} .$$

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2. On pose $u = \tan\frac{x}{2}$, $dx = \frac{2}{1+u^2}du$, $\cos x = \frac{1-u^2}{1+u^2}$, $\sin x = \frac{2u}{1+u^2}$

$$\Rightarrow I = \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1-u^2-4u} = \frac{1}{5}\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1-\left(\frac{u+2}{\sqrt{5}}\right)^2} \quad \text{or} \quad \frac{2}{1-v^2} = \frac{1}{1-v} + \frac{1}{1+v}$$

$$\text{d'où} \quad I = \frac{\sqrt{5}}{5}\left[\ln\left|\frac{1+\left(\frac{u+2}{\sqrt{5}}\right)}{1-\left(\frac{u+2}{\sqrt{5}}\right)}\right|\right]_{1/\sqrt{3}}^{1} = \frac{\sqrt{5}}{5}\left[\ln\left(\frac{1+\frac{3}{\sqrt{5}}}{1-\frac{3}{\sqrt{5}}}\right) - \ln\left|\frac{1+\frac{1/\sqrt{3}+2}{\sqrt{5}}}{1-\frac{\frac{1}{\sqrt{3}}+2}{\sqrt{5}}}\right|\right]$$

3. On pose $u = \ln x$ d'où $I = \displaystyle\int_{-\ln 2}^{0} \frac{u}{\sqrt{1-4u-u^2}}\,du$

$$= -\frac{1}{2}\int_{-\ln 2}^{0} \frac{-2u-4}{\sqrt{1-4u-u^2}}\,du \quad - \frac{2}{\sqrt{5}}\int_{-\ln 2}^{0} \frac{du}{\sqrt{1-\left(\frac{u+2}{\sqrt{5}}\right)^2}}$$

$$= \left[-\sqrt{1-4u-u^2} - 2\arcsin\left(\frac{u+2}{\sqrt{5}}\right)\right]_{-\ln 2}^{0} = -1 - 2\arcsin\Big(\underbrace{\frac{2}{\sqrt{5}}}_{<1\ !!}\Big) + \sqrt{1+4\ln 2-\ln^2 2} + 2\arcsin\left(\frac{2-\ln 2}{\sqrt{5}}\right)$$

Exercice 4.

$$\frac{x^3+1}{(x^2-4x+5)^2} = \frac{x(x^2-4x+5)+4x^2-5x+1}{(x^2-4x+5)^2} = \frac{x}{x^2-4x+5} + \frac{4(x^2-4x+5)+11x-19}{(x^2-4x+5)^2}$$

$$= \frac{x+4}{x^2-4x+5} + \frac{11x-19}{(x^2-4x+5)^2} = \frac{1}{2}\,\frac{2x-4}{x^2-4x+5} + \frac{6}{x^2-4x+5} + \frac{11}{2}\,\frac{2x-4}{(x^2-4x+5)^2} + \frac{3}{(x^2-4x+5)^2}$$

Donc :

$$\frac{x^3+1}{(x^2-4x+5)^2} = \frac{1}{2}\,\frac{2x-4}{x^2-4x+5} + \frac{11}{2}\,\frac{2x-4}{(x^2-4x+5)^2} + \frac{6}{1+(x-2)^2} + \frac{3}{(1+(x-2)^2)^2}$$

Pour intégrer cela, il ne nous manque que $\displaystyle\int\frac{du}{(1+u^2)^2}$. Or en posant $u = \tan\theta$ on a

$$\int\frac{du}{(1+u^2)^2} = \int\frac{d\theta}{1+\tan^2\theta} = \int\frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\arctan u}{2} + \frac{u}{2(1+u^2)}$$

On obtient donc enfin : $\longrightarrow$

$$\int \frac{x^3+1}{(x^2-4x+5)^2}\,dx = \frac{1}{2}\ln\left(x^2-4x+5\right) + \frac{11}{2}\left(\frac{-1}{x^2-4x+5}\right)$$

$$+ 6\arctan(x-2) + \frac{3}{2}\left(\arctan(x-2) + \frac{x-2}{1+(x-2)^2}\right) + K$$

$$= \frac{1}{2}\ln\left(x^2-4x+5\right) + \frac{15}{2}\arctan(x-2) + \frac{3x-17}{2(x^2-4x+5)} + K$$

Exercice 5.

- Supposons $\alpha \neq 1$. Alors $\displaystyle\int \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\, x^{1-\alpha}$

d'où $\displaystyle\int_a^X \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left(X^{1-\alpha} - a^{1-\alpha}\right) \longrightarrow +\infty$ si $1-\alpha > 0$

$\longrightarrow \dfrac{a^{1-\alpha}}{\alpha-1}$ si $1-\alpha < 0$.

donc $\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha}$ converge si $\alpha > 1$, diverge si $\alpha < 1$.

- pour $\alpha = 1$, $\displaystyle\int_a^X \frac{dx}{x} = \ln X - \ln a \xrightarrow[X \to +\infty]{} +\infty$

donc $\displaystyle\int_a^{+\infty} \frac{dx}{x}$ diverge.

Par conséquent, $\boxed{\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha} \text{ converge ssi } \alpha > 1.}$

De la même manière, exactement, on démontre que

$\boxed{\displaystyle\int_0^a \frac{dx}{x^\alpha} \text{ converge ssi } \alpha < 1.}$ $\Rightarrow$ Comme $\dfrac{1}{\sin^2 x} \geqslant \dfrac{1}{x^2}$

on a $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^2 x}$ diverge.